Ex.- Solve by the method of variation of parameters:

$$x^2 \frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$$

or, $y = c_1 \cos ax + c_2 \sin ax + \frac{\cos ax}{a^2} \log \cos ax + \frac{x}{a} \sin ax$

Sol. Dividing throughout by x^2 and writing it in normal form.

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2} y = x \quad \text{—— (I)}$$

By Inspection, $P + Qx = 0$, therefore $y = x$ is a part of CF. —— (II)

Again taking $y = vx$ in (I), we have

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = 0, \text{ where } u = x$$

or, $\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0$, The AE is $m^2 - 2m = 0 \Rightarrow m = 0, 2$

therefore $v = c_1 + c_2 e^{2x}$ —— (III)

Hence CF is $y = vx = c_1 x + c_2 xe^{2x}$ —— (IV)

Now let the sol. of (I) be $y = Ax + Bxe^{2x}$ —— (V)

where A and B are functions of x & are given by

$$\frac{dA}{dx}x + \frac{dB}{dx}xe^{2x} = 0 \quad \text{—— (VI)}$$

$$\frac{dA}{dx} + (1+2x)\frac{dB}{dx}e^{2x} = x \quad \text{—— (VII)}$$

Solve (VI) and (VII)

$$\frac{dA}{dx} = -\frac{1}{2} \quad \text{and} \quad \frac{dB}{dx} = \frac{1}{2}e^{-2x}$$

which on integration,

$$A = -\frac{x}{2} + K_1 \quad \text{and} \quad B = -\frac{1}{4}e^{-2x} + K_2$$

where K_1 and K_2 are arbitary constants of integration.

Hence CS is $y = (-\frac{1}{2}x + K_1)x + (-\frac{1}{4}e^{-2x} + K_2)xe^{2x}$

or, $y = K_1 x + K_2 x e^{2x} - \frac{1}{4}x - \frac{1}{2}x^2$ —— Ans.

Ex - Solve: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$ —— (I)

Sol. Taking RHS zero, the given eq. is homogeneous, so taking $x = e^z$ and writing $D = d/dz$, we have

$[D(D-1) + D - 1]\, y = 0 \Rightarrow (D^2 - 1)\, y = 0$ —— (II)

The AE is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

Therefore CF $y = c_1 e^z + c_2 e^{-z}$

$y = c_1 x + c_2 (1/x)$ —— (III)

Now let $y = Ax + B/x$, be the CS of (I) —— (IV)

where A & B are function of x. To determine these, choose A and B such that

$\frac{dA}{dx}x + \frac{dB}{dx}\frac{1}{x} = 0$ —— (V)

~~differentiating (IV) w.r. to x, we get~~

$\cancel{\frac{dy}{dx} = \frac{dA}{dx}.x + A + \frac{dB}{dx}}$

& $\frac{dA}{dx} \cdot 1 + \frac{dB}{dx}\left(-\frac{1}{x^2}\right) = e^x$ —(VI)

{ Here it is important that R.H.S. of above Eq.^{n} is e^x not $e^x . x^2$ as the value of R is determined after making the coffecient of $\frac{d^2y}{dx^2}$ as unity. }

$(V) + x.(vi) \Rightarrow 2x\frac{dA}{dx} = xe^x$

$\Rightarrow \frac{dA}{dx} = \frac{e^x}{2}$ ——(vii)

On Integration,

$\boxed{A = \frac{e^x}{2} + k_1}$ ——(viii)

put value of $\frac{dA}{dx}$ from (vii) in Eq.^{n}(V), we have $\frac{dB}{dx} = -\frac{1}{2}x^2e^x$

On Integration, we get

$B = -\frac{1}{2}[x^2e^x - 2xe^x + 2e^x] + k_2$

put value of A & B in Eq.^{n} (iv), ~~Eq.^{n}~~

required solution is

$y = (k_1 + \frac{1}{2}e^x)x + [-\frac{1}{2}\{x^2e^x - 2xe^x + 2e^x\} + k_2]\frac{1}{x}$

Ex:- Solve : $\frac{d^2y}{dx^2} + \underbrace{(1-\cot x)}_{P}\frac{dy}{dx} - y\cot x = \underbrace{\sin^2 x}_{R}$ —(i)

Sol^n: → Here $1 - P + Q = 0$ [as $Q = -\cot x$]

$\Rightarrow \boxed{y = e^{-x}}$ is a part of C.F. $\rightarrow u$

In order to determine C.F., let

$y = e^{-x}.v$ ——(ii)

put this in eq.^{n} (i) with consideration $R = 0$, we have

$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{0}{u}$ as $R = 0$

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$$\Rightarrow \frac{d^2v}{dx^2} + \left[1 - \cot x + \frac{2}{e^{-x}}(-e^{-x})\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} + (-1-\cot x)\frac{dv}{dx} = 0 \quad \text{—(iii)}$$

let $\frac{dv}{dx} = p$

$$\Rightarrow \frac{dp}{dx} + (-1-\cot x)p = 0$$

$$\Rightarrow \frac{dp}{p} = (1+\cot x)\,dx$$

On Integration

$$\Rightarrow \log p = x + \log \sin x + \log c_1$$

$$\Rightarrow \log p = \log(c_1 \sin x\, e^{x})$$

$$\Rightarrow p = c_1 \sin x\, e^{x} = \frac{dv}{dx}$$

On Integration.

$$\Rightarrow v = c_1\left[\frac{e^x}{1^2+1^2}(\sin x - \cos x)\right] + c_2$$

$$\Rightarrow v = c_1\left[\frac{e^x}{2}(\sin x - \cos x)\right] + c_2$$

Hence $y = ve^{-x}$

$= \frac{1}{2}c_1(\sin x - \cos x) + c_2 e^{-x}$ is

C.F. of given Equation.

[Now to determine G.S., Replace

$c_1 \to A$, $c_2 \to B$ in C.S.]

(23)

let $y = A\left[\underbrace{\frac{1}{2}(\sin x - \cos x)}_{u}\right] + B\underbrace{e^{-x}}_{v}$ — (✱)

is CS of given DE, where A & B are funⁿ of x s.t.

$$\frac{1}{2}(\sin x - \cos x)\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \quad \text{— (iv)}$$

$$\frac{1}{2}(\cos x + \sin x)\frac{dA}{dx} - e^{-x}\frac{dB}{dx} = \sin^2 x \quad \text{— (V)}$$

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eqⁿ (iv) + (V) ⇒ $\sin x\frac{dA}{dx} = \sin^2 x \Rightarrow \frac{dA}{dx} = \sin x$ — (vi)

On integration $\boxed{A = -\cos x + k_1}$ — (vii)

substitute the value of $\frac{dA}{dx}$ in (iv), we have.

$$\frac{dB}{dx} = -\frac{e^{x}\sin x(\sin x - \cos x)}{2} = \frac{1}{2}e^{x}(-\sin^2 x + \sin x\cos x)$$

$$\frac{dB}{dx} = \frac{e^x}{2}\left[\frac{\sin 2x}{2} - \frac{(1-\cos 2x)}{2}\right]$$

$$\frac{dB}{dx} = \frac{e^x}{4}[\sin 2x + \cos 2x - 1]$$

On ~~bot~~ integration

$$B = \frac{1}{4}\left[\frac{e^x}{1^2+2^2}\{1.\sin 2x - 2\cos 2x\} + \frac{e^x}{1^2+2^2}\{1.\cos 2x + 2\sin 2x\} - e^x\right] + k_2$$

$$\Rightarrow B = \frac{1}{20}\left[e^x(3\sin 2x - \cos 2x)\right] - \frac{e^x}{4} + k_2$$

Put value of A & B in (✱) will give CS.